MATH 2040A Tutorial 5th Week

4.1 Determinants of Order 2

Recall some basic facts about determinants of order 2.

5. If B is the matrix obtained by interchanging the rows of a 2×2 matrix A, then $\det B = -\det A$.

6. If the 2 columns of $A \in M_{2\times2}(F)$ are identical, then $\det A = 0$ (F can be taken as $\mathbb{R}$ or $\mathbb{C}$.)

7. For any $A \in M_{2\times2}(F)$ $\det A^t = A$.

8. If $A = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \in M_{2\times2}(F)$, then $\det A = \prod_{i=1}^{n} a_i$

9. For any $A, B \in M_{2\times2}(F)$, $\det AB = \det A \cdot \det B$

Similar properties hold for matrices in $M_{n\times n}(F)$.

For example, we prove 7 for 2×2 matrix A.

$$\det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - cb = ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

4.3 Properties of determinants

10. $M \in M_{n\times n}(\mathbb{C})$. M is called nilpotent if for some positive integer k, $M^k = 0$, where O is the $n\times n$ zero matrix.

Prove that if M is nilpotent then $\det M = 0$

Solution: $\exists k \in \mathbb{Z}_{>0}$ s.t. $M^k = \mathcal{O} \Rightarrow \det(M^k) = 0 \Rightarrow$

$(\det M)^k = 0 \Rightarrow \det M = 0$. $\uparrow$ by basic properties of determinants.

12. $Q \in M_{n\times n}(\mathbb{R})$ is called orthogonal if $QQ^t = I$. Prove that if Q is orthogonal, then $\det Q = \pm 1$.

Solution: $QQ^t = I \Rightarrow \det Q \det Q^t = \det I = 1$

by basic properties!

$\Rightarrow (\det Q)^2 = 1 \Rightarrow \det Q = \pm 1$.

Remark: You can solve Ex 15 similarly.

22. $T : P_n(F) \to F^{n+1}$ linear transformation

$T(f) = (f(c_0), f(c_1), \dots, f(c_n))$ c_i are distinct scalars in an infinite field F.

β - standard ordered basis for $P_n(F)$

γ - ... for F^{n+1}

a) Show that $M = [T]_\beta^\gamma = \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & & & \\ \vdots & \vdots & & & \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}$ $\leftarrow$ Vandermonde Matrix.

b) $\det M \neq 0$.

c) $\det M = \prod_{0 \le i < j \le n} (c_j - c_i)$

Solution (1) $T(1) = 1 \cdot e_1 + 1 \cdot e_2 + \cdots + 1 \cdot e_{n+1}$

$$T(x) = c_0 \cdot e_1 + \cdots + c_n \cdot e_{n+1}$$

$$\vdots$$

$$T(x^n) = c_0^n e_1 + \cdots + c_n^n e_{n+1}$$

$\{e_i\}_{i=1 \dots n}$ is the standard basis for $\mathbb{F}^{n+1}$.

(2) T is an isomorphism. $\Rightarrow \det [T]_\beta^\gamma \neq 0$

(Why? T is injective linear map and domain ~~of~~ and codomain of T is of the same dim.

We now prove that T is indeed injective:

$T(f) = 0 \Rightarrow (f(c_0) \cdots f(c_n)) = 0 \Rightarrow c_i$ are roots of f. But f is a polynomial of deg n with $n+1$ distinct roots $\Rightarrow f = 0$. i.e. T is injective)

(3) We can prove this by induction on n.

Details are presented in class.

5.1 Eigenvalues And Eigenvectors

11. A scalar matrix is a square matrix of the form λI for some sclar λ : that is, a scaler matrix is a diagonal matrix in which all the diagonal entries are equal.

a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.

b) A diagonalizable matrix having only one eigenvalue is a scalar matrix.

c) Prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Solution: a) $A = B^{-1} \lambda I B = \lambda I$ B invertible.

b) We have a basis $\{v_i\}$ s.t. $(M - \lambda I) v_i = 0$, under which, M is diagonal.

$\Rightarrow (M - \lambda I) v = 0$ for every v ($\{v_i\}$ is a basis).

$\Rightarrow M = \lambda I$.

c) 1 is the only eigenvalue of the matrix.

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ whose nullity is 1.

$\Rightarrow$ We cannot find a set of two vectors consisting of eigenvectors s.t. the set is independent.

$\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

12. Prove that similar matrices have the same characteristic polynomial

Solution If $A = P^{-1}BP$, then

$$\det(A - \lambda I) = \det(P^{-1}(A - \lambda I)P) = \det(P^{-1}AP - \lambda P^{-1}IP)$$
$$= \det(B - \lambda I)$$

Remark: Try to show that the definition of the characteristic polynomial of a linear operator on a fin. dim. v.s. V is independent of the choice of basis for V (~~tt~~ i.e. char. polynomial is well defined).

14. For any square matrix A, prove that A and A^t have the same characteristic polynomial (hence the same eigenvalues)

Solution: $\det(A - \lambda I) = \det((A - \lambda I)^t) = \det(A^t - \lambda I)$

15. T linear operator on a vector space V
x eigenvector of T with eigenvalue λ
For any positive integer m, prove that x is an eigenvector of T^m corresponding to the eigenvalue λ^m

Solution: $Tv = \lambda v$

$$T^m v = T^{m-1} \lambda v = \cdots = \lambda^m v.$$